DUOPOLY MODELS*

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1 Collusion in Duopoly

Let us suppose that a market has the following inverse demand function:

$$P = 100 - 0.5(q_1 + q_2) \tag{1}$$

There are two firms in this market. They have the following cost curves:

$$C_1 = 5q_1 \tag{2}$$
$$C_2 = 0.5q_2^2 \tag{3}$$

If the two firms can collude, they will maximize the joint profit by acting as a *de facto* monopolist. The joint profit function is given by

$$\Pi = [100 - 0.5(q_1 + q_2)](q_1 + q_2) - 5q_1 - 0.5q_2^2 \tag{4}$$
$$= 100(q_1 + q_2) - 0.5(q_1 + q_2)^2 - 5q_1 - 0.5q_2^2 \tag{5}$$
$$= 100(q_1 + q_2) - 0.5(q_1^2 + 2q_1q_2 + q_2^2) - 5q_1 - 0.5q_2^2 \tag{6}$$

The two firms choose q_1^* and q_2^*, i.e., the values of q_1 and q_2 that maximize the joint profit. The two first order conditions are given by

$$\frac{\partial \Pi}{\partial q_1} = 100 - q_1 - q_2 - 5 = 0 \tag{7}$$
$$\frac{\partial \Pi}{\partial q_2} = 100 - q_1 - q_2 - q_2 = 0 \tag{8}$$

Solving the two first order conditions simultaneously, we should get $q_1^* = 90$ and $q_2^* = 5$. Now that we know the value of $(q_1 + q_2)$, we should get $P^* = 52.5$. Finally, given the optimum quantities and the price, we should be able to compute the profits earned by the two companies in this market with collusion: $\Pi_1^* = 4275$ and $\Pi_2^* = 250$.

2 Cournot Competition

Suppose now that the firms cannot collude and are involved in Cournot competition. In that case, they will not jointly maximize profit; each form will individually maximize its own profit and choose the corresponding optimal level of output. The profit functions of the two firms are given by

$$\Pi_1 = [100 - 0.5(q_1 + q_2)]q_1 - 5q_1 \tag{9}$$

$$\Pi_2 = [100 - 0.5(q_1 + q_2)]q_2 - 0.5q_2^2 \tag{10}$$

The first order conditions for profit maximization are

$$\frac{\mathrm{d}\Pi_1}{\mathrm{d}q_1} = 100 - q_1 - 0.5q_2 - 5 = 0 \tag{11}$$

$$\frac{\mathrm{d}\Pi_2}{\mathrm{d}q_2} = 100 - 0.5q_1 - q_2 - q_2 = 0 \tag{12}$$

Equation (11) is the reaction function for Firm 1, while equation (12) is the reaction function for Firm 2. We know that in a market with Cournot competition involving two firms, the equilibrium is attained at the point where the two reaction functions intersect. Hence, we solve equations (11) and (12) simultaneously, and get $q_1^* = 80$ and $q_2^* = 30$. Given the optimum quantities, we can compute the price in equilibrium: $P^* = 45$. Finally, given the quantities and the price in equilibrium, we can compute the profits of the two firms: $\Pi_1^* = 3200$ and $\Pi_2^* = 900$.

Note the following:

- The total amount produced under Cournot competition (110) is higher than the total amount produced when the two firms collude (95).

- Not surprisingly, the price is lower under Cournot competition (45) than in a market with collusion (52.5).

- In the market with Cournot competition, more is produced in the firm with the convex cost curve (30) than in the market with collusion (5), while less is produced in the firm with the linear cost curve.

- Not surprisingly, the total profit earned by the two firms is lower in a market with Cournot competition (4100) than in a market with collusion (4525).

3   Cournot Competition when One Firm is Subsidized

Next, suppose that we still have Cournot competition in the market, but Firm 1 is subsidized by the government. The extent of subsidy is £s per unit of output produced by the firm. Firm 2 is not subsidized. In that case, Firm 1's cost function becomes $C_1' = 5q_1 - sq_1 = (5 - s)q_1$. Hence, Firm 1's profit function becomes

$$\Pi_1' = [100 - 0.5(q_1 + q_2)]q_1 - (5 - s)q_1 \tag{13}$$

The new reaction function of Firm 1, therefore, is given by

$$\frac{\mathrm{d}\Pi_1'}{\mathrm{d}q_1} = 100 - q_1 - 0.5q_2 - 5 + s = 0 \tag{14}$$

The profit function of Firm 2 is unchanged and hence its reaction function does not change either. The Cournot equilibrium is obtained by simultaneously solving equations (12) and (14), and it gives us $q_1^* = \frac{560 + 8s}{7}$ and $q_2^* = \frac{105 - s}{3.5}$. You can see that subsidy to Firm 1 increases Firm 1's output and decreases Firm 2's output.

4   Stackelberg Duopoly

Let us now move back to a duopoly market where neither firm receives a subsidy. But now Firm 1 acts as the leader and Firm 2 as the follower. This is the Stackelberg paradigm. In this paradigm, Firm 1, the leader, knows how Firm 2 will react to its output decision, i.e., Firm 1 knows Firm 2's reaction function.

We know that equation (12) gives us Firm 2's reaction function. Simplifying it, we get $q_2 = 50 - 0.25q_1$. In the Stackelberg paradigm, Firm 1's profit function, therefore, is given by

$$
\begin{aligned}
\Pi_1'' &= [100 - 0.5(q_1 + q_2)]q_1 - 5q_1 & (15)\\
&= 100q_1 - 0.5q_1^2 - 0.5q_1q_2 - 5q_1 & (16)\\
&= 100q_1 - 0.5q_1^2 - 0.5q_1(50 - q_1) - 5q_1 & (17)\\
&= 70q_1 - 0.375q_1^2 & (18)
\end{aligned}
$$

Firm 1's first order condition for profit maximization, therefore, is

$$\frac{\mathrm{d}\Pi_1''}{\mathrm{d}q_1} = 70 - 0.75q_1 = 0 \tag{19}$$

We, therefore, get $q_1^* = 93.33$ and, substituting this value of q_1 in the reaction function of Firm 2, we get $q_2^* = 50 - 0.25q_2 = 26.67$. Given these values of output, we can once again compute the price and the profits of the two firms in equilibrium.

The profit of Firm 1 (the leader) is 3266.67, higher than the firm's profit in a market with simple Cournot competition. Correspondingly, the profit of Firm 2 (the follower) is 711.11, lower than the firm's profit in a market with simple Cournot competition.

5 Hotelling Model[1]

Let there be two retail outlets, 1 and 2, that are located along a road that is l kilometers long. At each point along this road, there is a consumer who can choose to go to either outlet 1 or outlet 2, but they have to bear the cost of travel along this road. The cost is £c per unit of length. Each retail outlet sets a price that maximizes its profits; outlet 1 charges price p_1 while outlet 2 charges price p_2. The diagramatic representation of this arrangement is as follows:

$-----a------\mathbf{1}-----x------ \downarrow ------y------\mathbf{2}-----b------$
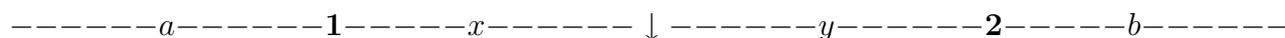
It is easy to see that consumers who live in zone a are too far away from outlet 2 and it will always be cheaper for them to go to outlet 1. Similarly, for consumers who live in zone b, it is always cheaper to go to outlet 2. In other words, outlet 1 has monopoly over consumers in zone a while outlet 2 has monopoly over consumers in zone b. For those consumers who live between outlets 1 and 2, the total cost of purchasing the product from outlet 1 has to be compared with the total cost of purchasing the product from outlet 2. There would necessarily be one consumer who would be indifferent between the two outlets. She would be located at the point marked by $\downarrow$, that is at a distance x from outlet 1 and at a distance y from outlet 2.

In other words, the following two conditions should hold:

$$l = a + x + b + y \tag{20}$$
$$p_1 + cx = p_2 + cy \tag{21}$$

We can solve equations (20) and (21) simultaneously to obtain

$$x = \frac{1}{2}\left(l - a - b + \frac{p_2 - p_1}{c}\right) \tag{22}$$
$$y = \frac{1}{2}\left(l - a - b + \frac{p_1 - p_2}{c}\right) \tag{23}$$

An increase in p_1 reduces the value of x and increases the value of y, i.e., if outlet 1 increases its price then fewer people who live between the two outlets find it cheaper to go to outlet 1, while more people find it cheaper to go to outlet 2. An increase in p_2 has the opposite effect.

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[1]Harold Hotelling (1929). Stability in competition. *Economic Journal*, Vol. 39, No. 153, pp. 41-57.

Let us assume for the sake of simplicity that the cost of each retail outlet is zero. Given than there is one consumer per unit of the road, it is easily seen that the demand experienced by the two outlets are $q_1 = (a + x)$ and $q_2 = (b + y)$. In the absence of cost, the profit functions of the outlets are

$$\Pi_1 \quad = \quad p_1 q_1 = p_1(a + x) \tag{24}$$

$$\Pi_2 \quad = \quad p_2 q_2 = p_2(b + y) \tag{25}$$

Substituting the values of x and y from equations (22) and (23) into equations (24) and (25), respectively, we get

$$\Pi_1 \quad = \quad \frac{1}{2}(l + a - b)p_1 - \frac{p_1^2}{2c} + \frac{p_1 p_2}{2c} \tag{26}$$

$$\Pi_2 \quad = \quad \frac{1}{2}(l - a + b)p_2 - \frac{p_2^2}{2c} + \frac{p_1 p_2}{2c} \tag{27}$$

The first order conditions of profit maximization of the two outlets are given by

$$\frac{\mathrm{d}\Pi_1}{\mathrm{d}p_1} \quad = \quad \frac{1}{2}(l + a - b) - \frac{p_1}{2c} + \frac{p_2}{2c} = 0 \tag{28}$$

$$\frac{\mathrm{d}\Pi_2}{\mathrm{d}p_2} \quad = \quad \frac{1}{2}(l - a + b) + \frac{p_1}{2c} - \frac{p_2}{2c} = 0 \tag{29}$$

We can simultaneously solve equations (28) and (29) to obtain

$$p_1^* \quad = \quad c\left(l + \frac{a - b}{3}\right) \tag{30}$$

$$p_2^* \quad = \quad c\left(l - \frac{a - b}{3}\right) \tag{31}$$

Given the optimum prices, we can compute the optimal values of profits of the two outlets

$$\Pi_1^* \quad = \quad \frac{c}{2}\left(l + \frac{a - b}{3}\right)^2 \tag{32}$$

$$\Pi_2^* \quad = \quad \frac{c}{2}\left(l + \frac{b - a}{3}\right)^2 \tag{33}$$

It can be seen that p_1^* and Π_1^* increase with a while p_2^* and Π_2^* increase with b, i.e., the price that an outlet can charge and its profit increase with the size of the pool of consumers over which it has monopoly. Hence, it would be rational for each outlet to try to maximize the pool of consumers over which it has monopoly; outlet 1 would move towards outlet 2, and vice versa. In equilibrium, therefore, outlets 1 and 2 would be next to each other somewhere in the middle of the street.